SURVEY OF INDIA

PROFESSIONAL PAPER No. 22.

THREE SOURCES OF ERROR IN PRECISE LEVELLING

BY

CAPTAIN G. BOMFORD, R.E.

PUBLISHED BY ORDER OF
BRIGADIER E. A. TANDY
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INTRODUCTION

When large circuits of geodetic levelling are completed, they are generally found to close with an error larger than would have been expected from the accumulation of the usual accidental error. This fact is expressed by ascribing some source of systematic error to the work, and the International Standard of high precision levelling allows such a systematic error of 0·00106 feet per mile. Nevertheless the presence of systematic error in any work is unsatisfactory, being in fact an admission of failure to devise a perfect system.

Apart from movements of the Earth's crust there seem to be four ways in which large errors may arise in our present system of high precision levelling :—

(1) Movements of the peg or bench-mark on which the staff is placed: either during work, during the night, or during the months which elapse between successive seasons. Colonel Burrard* has laid emphasis on this source of error, and precautions against it now take the form of avoiding railways, of levelling the line once in each direction, and in seeking to close the season's work on rock cut bench-marks. The last especially is often impossible, and movements of pegs and bench-marks must still be considered a possible source of systematic error.

(2) Wooden levelling staves are imperfect instruments. Irregular expansion may lead to small but appreciable errors in different parts of the staff after correction for overall length. When levelling uphill one part of the staff may always be read in the back position, and a different part in the fore position, so that a systematic error may result. It is expected that invar staves will shortly be introduced into the Survey of India, and this source of error, which is in any case unlikely to be serious, should then disappear.

(3) Irregular refraction may introduce errors of some thousandths of a foot at a single station. Under ordinary circumstances they will not accumulate, but when working on a long continuous gradient they would tend to do so.

(4) The crossing of wide unbridged rivers involves a great loss of accuracy.

This paper contains the results of enquiries into three points connected with these sources of error viz :—

Part I. The crossing of unbridged rivers.
Part II. The error due to refraction when levelling up a hill.
Part III. The correction for staff length.

* G.T.S. Vol. XIX pages 353, 354.

The first investigation was undertaken partly to test the merits of two levels which had been fitted with eye-piece micrometers for the measurement of small vertical angles, and partly to discover what accuracy it is in fact possible to obtain when crossing a wide river. The latter point is one on which considerable doubt exists. For instance, probable errors have been deduced from the accordance of the various individual measures made, and have generally been recorded as less than 0·010 feet; on the other hand, when a circuit has closed with some large error, such as a whole foot, the river crossings contained in it have been suspected of causing the error. A formula for the probable error has now been deduced, depending on the width of the river and other conditions. It pretends to very little accuracy, but it is believed to give a useful indication. It has also been concluded that rivers of 30 chains or more in width should be crossed at several points along the banks, a short time being spent at each, rather than several days being spent at one place. This is on account of the persistence of sign of the errors found at individual sites. A further trial of the water-gauge method is also recommended.

Of the appendices to this part, A, C and G refer to points which are not novel but which are apt to be overlooked. B, D and E give detailed results of the experiments, the interest of the last two lying largely in the figures given for the temperature gradient. Appendix F gives some formulæ for the curvature and coefficient of refraction in horizontal rays of light.

Part II contains the deduction of the refraction correction in the line from Dehra Dūn to Mussoorie, based on observations made in 1927. The correction is found to be of about the same order of magnitude as the usual probable error of a line of the same length. It is, however, so uncertainly determined that it is not considered worth applying to the observed heights. Although the correction is almost negligible in this case, the formula shows that it may sometimes be of consequence on a line which rises less steeply. It is thought that this is more satisfactorily dealt with by reducing the length of shot on these rare occasions, than by trying to deduce a correction from temperature readings. Rules are given which indicate when such reduction should be made.

Part III enquires into a point recently raised by Dr. J. de Graaff Hunter, namely that the present method of applying the staff correction is only exact if the staves are of equal length. It is concluded that this method suffices, provided the inequality does not exceed 0·004 feet, a standard which can be maintained without much difficulty.

LEVELLING ACROSS UNBRIDGED RIVERS

Fig. 1

Balawali Ray AX

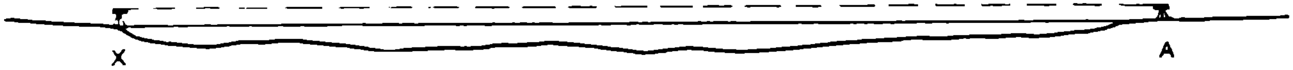


Fig. 2

Balawali Ray BY

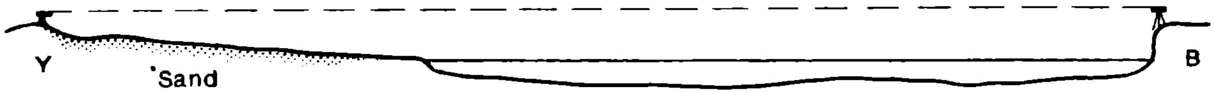


Fig. 3

Balawali Ray CZ

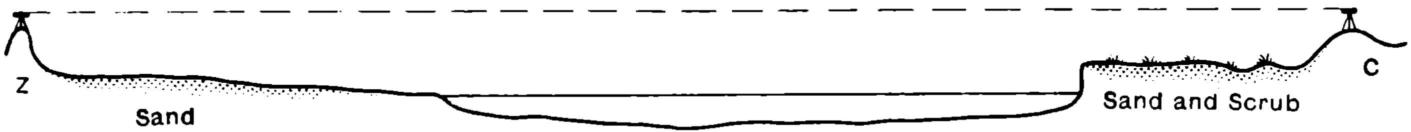
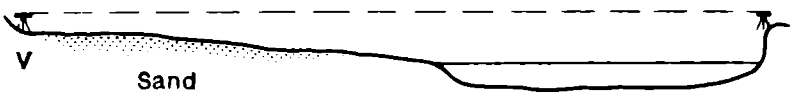


Fig. 4

Balawali Ray DV



Scale of Figures.—Horizontal 900 feet=1 inch

Vertical 60 feet=1 inch

For dimensions see Table 7

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LEVELLING ACROSS UNBRIDGED RIVERS

Fig. 5 Balawali Ray EW

Fig. 6 Damukdia

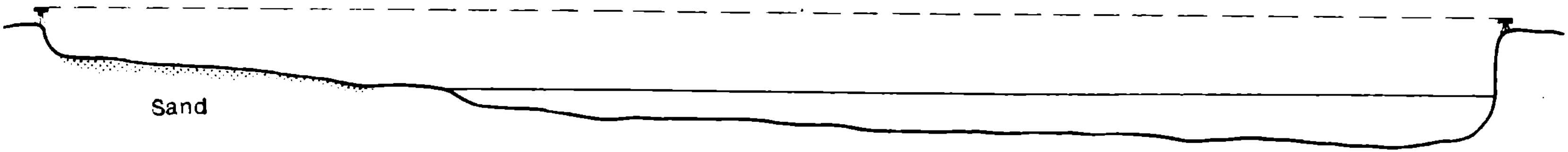

Fig. 7 Typical unsymmetrical crossing

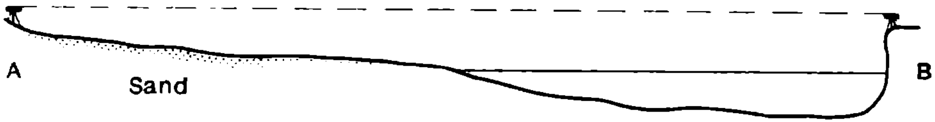

Fig. 8 Use of 3 Levels

Scale of Figures.— Horizontal 900 feet=1 inch

Vertical 60 feet=1 inch

For dimensions see Table 7

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PART I

LEVELLING ACROSS UNBRIDGED RIVERS

Introductory.—An unbridged river a quarter of a mile or more in width is an obstacle in the course of ordinary levelling, the overcoming of which not only requires special methods of work, but also results in serious loss of accuracy. Three methods have previously been tried, viz :-

(A) Water-gauge.
(B) Moving target and level.
(C) Vertical angles with a theodolite.

These three methods are described by Capt. H. L. Crosthwait, R.E. in Survey of India Professional Paper No. 7 of 1903, with reference to a crossing made by him at Damukdia, Bengal. The water-gauge method has lately been considered unreliable. The target method is slow and laborious, and also liable to error on account of bias acquired by the man operating the target. The vertical angle method requires two large theodolites, the use of which is beyond the powers of ordinary levelling personnel. As an alternative Lieut.-Col. V. R. Cotter has had two American binocular levels fitted with micrometer eye-pieces. Like theodolites they can be sighted on to a fixed target, while their use requires no more skill than is necessary for ordinary levelling.

In 1927 five experimental river crossings were made with these levels by simultaneous reciprocal observations, near the Ganges bridge at Balawali (U.P.), where the true difference of height could be found by ordinary levelling over the bridge. Details of the five rays are given in Figs. 1 to 5 and Table 7.

The levels were found entirely satisfactory, except that, since they are not reversible, it is necessary to determine their collimation error with a troublesome degree of accuracy. The use of these instruments is recommended for future work; or, if possible, the use of a reversible level (such as Zeiss') similarly fitted.

With reasonable care serious instrumental error can probably be eliminated from any of the last three methods :—target, theodolite, and micrometer level. But all rely equally on the assumption that the angles of refraction at either end are equal. The Balawali experiments were principally carried out with a view to determining what sized errors are likely to arise from this assumption, and what siting of the ray is most favourable. It is obvious that great length and an unsymmetrical site have adverse effects, while increased height above the ground or water is beneficial. At least two of these factors (height and shortness) are antagonistic, so that the choice of the best possible site is inevitably a difficult matter of judgment. The following brief statement of the causes of irregular refraction, and of the probable effects of length, height and asymmetry, is intended to serve as a guide by which an opinion may be formed regarding the relative merits of different sites.

Refraction.—Let the distance between the two stations A and B be a feet.

Let the angles of refraction at A and B be Ω_A and Ω_B seconds respectively.

Let the coefficients of refraction deduced from the observations at A and B be η_A and η_B respectively,

$$\text{i.e. } \eta_A = \frac{100\Omega_A}{a} \text{ approximately.}$$

It may be shown (See Appendix F) that the curvature of a horizontal ray of light passing through air in which the isothermal layers are reasonably horizontal, varies as $\left(0\cdot0187 + \frac{dT}{dh}\right)$, where T is the Fahr. temperature, and h is the height in feet, $\frac{dT}{dh}$ being the rate of increase of temperature with height (usually negative). With ordinary sea-level values of temperature and pressure the resulting coefficient of refraction is $5\cdot4\left(0\cdot0187 + \frac{dT}{dh}\right)$, and local irregularities in it are caused by differences in $\frac{dT}{dh}$.

For example, the normal value of $\frac{dT}{dh}$, undisturbed by the proximity of ground or water is 3° F per 1000 feet, i.e. $-0\cdot003$. This gives $\eta = +0\cdot085$. But within 20 feet of the ground the gradient is often as much as 60° F per 1000 feet, giving $\eta = -0\cdot222$. And over cold water the temperature may increase with height at a similar rate, giving $\eta = +0\cdot417$. Such gradients are in no way unusual. (See Appendix E).

If reciprocal angles are observed, a large or small value of η does not in itself cause error. Error arises if it is systematically different at opposite ends of the ray. For the angle of refraction at one end of the ray varies as the integral, with respect to distance along the ray, of the product of the curvature at any point with the distance of that point from the far end. In symbols, $\Omega_A = \frac{1}{a}\int_0^a (\text{curvature})\,(a-x)\,dx$. It is clear that Ω_A is principally influenced by the conditions obtaining near A, and is affected to a much lesser degree by conditions near B.

The error in height difference arising from the assumption that $\Omega_A = \Omega_B$ is clearly $\frac{a}{2}\left(\Omega_A - \Omega_B\right)\sin 1''$. The following example will serve to show the kind of error which will arise in a crossing which is half over sand and half over water. (See Fig. 7). $a = 4000'$. Assume the gradient over sand to be $-30°$ F per 1000 feet, and over water $+30°$ F per 1000 feet. The corresponding curvatures are $-0\cdot00122$ and $+0\cdot00526$ seconds per foot respectively.

$$\text{Then } \Omega_A = \frac{1}{4000}\int_0^{2000}(-0\cdot00122)\ (4000-x)dx + \frac{1}{4000}\int_{2000}^{4000}(+0\cdot00526)\ (4000-x)\ dx = +0\cdot79''$$

$$\text{And } \Omega_B = \frac{1}{4000}\int_0^{2000}(+0\cdot00526)\ (4000-x)dx + \frac{1}{4000}\int_{2000}^{4000}(-0\cdot00122)\ (4000-x)\ dx = +7\cdot29''$$

Then the error will be $2000 \times 6\cdot5 \sin 1'' = \cdot063$ feet.

Its sense will be such as to make the height of B above A too small.

In practice it is not possible to measure the temperature gradients, and so to apply a correction of this nature, and the above example is only intended to show the kind of error which may be expected under unsymmetrical conditions.

Comparison of different sites.—Sources of error may be considered under two headings:—

(1) Error due to visible asymmetry, such as a preponderance of sand on one bank. The sign of this can be determined, and a guess can be made at its magnitude.

(2) Casual error, and that due to inconspicuous asymmetry, such as a hot wind blowing off one bank. An estimate of this error can only be expressed in the form of a probable error depending on the length of the ray, and its height above the ground or water.

Taking the second source first, the effect of length may be considered on three hypotheses:—

(1) That the error arises only at the banks, being due to shallow water, wind off the land, or similar causes. In this case $\Omega_A - \Omega_B$ is independent of length, and the error in height will vary directly as the length a.

(2) That casual irregularities occur throughout the ray. $\Omega_A - \Omega_B$ then varies as $a^{\frac{1}{2}}$ and the height error varies as $a^{\frac{3}{2}}$.

(3) That there is a marked but unnoticed asymmetry, such as one half of the river being systematically at a different temperature to the other. In this case $\Omega_A - \Omega_B$ varies as a. Height error varies as a^2.

Case (2) seems the most probable, and also strikes a mean between the others. The statement that the error will vary as $(\text{length})^{\frac{3}{2}}$ is likely to be fairly close to the truth.

The effect of height is more doubtful. Three different hypotheses may again be considered, and the results combined.

(1) It is reasonable to suppose that the probable error of a symmetrical crossing of a given length at a certain height will vary

as the magnitude of the temperature gradients generally met with at that height.

The following table is abstracted from Appendix E.

TABLE 1

	Balawali 8·00 to 11·00 hours	Balawali 11·00 to 16·00 hours	Balawali Mean	Dehra Dūn 11·00 to 16·00 hours
Gradient at 13 feet in °F per 1000 feet	−117	−124	−120	−78
„ „ 31 feet „ „ „	−15	−31	−23	−16
Ratio ..	...	...	5·2	4·9

On these grounds a crossing at 31 feet might be considered five times as reliable as one at 13 feet.

A similar table cannot be formed with the gradients deduced from the observed refractions in the Balawali river crossings, because at a certain height above water (perhaps at 20 feet or 30 feet) the positive reverse gradient changes to a normal negative one, giving a mean of zero. But the fallibility of such a ray is not zero.

(2). Errors being due to changes of gradient from place to place, it may be assumed that the probable errors at different heights are proportional to the variability with time of the gradients measured at a single place. Inspection of the first two columns of Table 18 (Appendix E) shows that the measures at 13 feet vary through a wider range than do those at 31 feet. The amount of variation is conveniently expressed by the Standard Deviation or root mean square of the departures from the mean value.

The following table is abstracted from Appendix E.

TABLE 2

	Balawali 8·00 to 11·00 hours	Balawali 11·00 to 16·00 hours	Balawali Mean	Dehra Dūn 11·00 to 16·00 hours
Standard Deviation at 13 feet in °F per 1000 feet	45	48	46	44
„ „ „ 31 feet „ „ „	29	26	27	20
Ratio ...	...	...	1 7	2·2

On these grounds a crossing at 31 feet might be considered twice as reliable as one at 13 feet.

Table 3 shows the extreme range of variation in the gradients deduced from the different Balawali crossings. As it would be unsatisfactory to compare rays which are largely over sand with those which are over water, only rays which are predominantly over water have been included.

TABLE 3

Station of observation	Mean height of ray (feet	Range of variation of gradients (°F)	Ratio compared with B
A	5	245 } 209	3·54
X	5	172 }	
B	14	59	1·00
C	22	36	0·61
D	12	100	1 69

(3). The range of variation of the hourly measures of the height difference found in the different Balawali crossings is also a measure of the fallibility of rays at different heights. The effect of asymmetry is got rid of by considering variation only, not absolute error. It is necessary to reduce all the rays to a common length by dividing by a factor proportional to $(\text{length})^{\frac{3}{2}}$.

Table 4 is abstracted from Tables 13 to 17 (Appendix D).

TABLE 4

Ray	Length (feet)	Mean Height (feet)	Range of variation of height difference				Mean ÷ $\left(\frac{\text{length}}{3300}\right)^{\frac{3}{2}}$	Ratio compared with BY
			1st day	2nd day	3rd day	Mean		
			feet	feet	feet	feet		
AX	4650	5	0·208	0·218	0·115	0·180	0·108	1·15
BY	4980	14	0 129	0·217	0·176	0·174	0·094	1·00
CZ	5940	22	0·119	0·273	0·321	0·238	0·098	1·04
DV	3300	12	0·138	0·113	...	0·125	0·125	1 33
EW	4170	22	0 161	0·170	...	0·166	0·117	1·24

On this evidence the reliability might be considered to be independent of the height. The data are both reasonably full and reasonably consistent, and it is surprising that they should give such a result.

A similar table may be formed with the range of variation of the daily means.

TABLE 5

Ray	Length (feet)	Mean height (feet)	Range of variation (feet)	Range ÷ $\left(\frac{\text{length}}{3300}\right)^{\frac{3}{2}}$	Ratio compared with BY
AX	4650	5	0·062	0·037	1·48
BY	4980	14	0·047	0·025	1·00
CZ	5940	22	0·028	0·012	0·48

DV and EW have been excluded as only two days' observations were made.

Table 6 summarises the previous five tables, and means the results, giving each equal weight. For convenience, the results for 13 feet in the first two tables have been combined with those for 14 feet in the others, and attributed to $13\frac{1}{2}$ feet.

TABLE 6

Height (feet)	Fallibility according to					Mean
	Table 1	Table 2	Table 3	Table 4	Table 5	
5	...	...	3·54	1·15	1·48	2·06
12	...		1·69	1·33	...	1·51
$13\frac{1}{2}$	1·00	1·00	1·00	1·00	1·00	1·00
22	...	...	0·61	1·14	0·48	0·74
31	0·20	0·50	...	...	...	0·35

These results are plotted in Fig. 9. They are well fitted by the curve Fallibility $= 2 \cdot 9e^{-\cdot 065h}$, and it will henceforward be assumed that the probable error of a ray varies as $e^{-\cdot 065h}$, where h is its average height in feet.

Then the probable error $= A\,a^{\frac{3}{2}}e^{-\cdot 065h}$, where A is a constant.

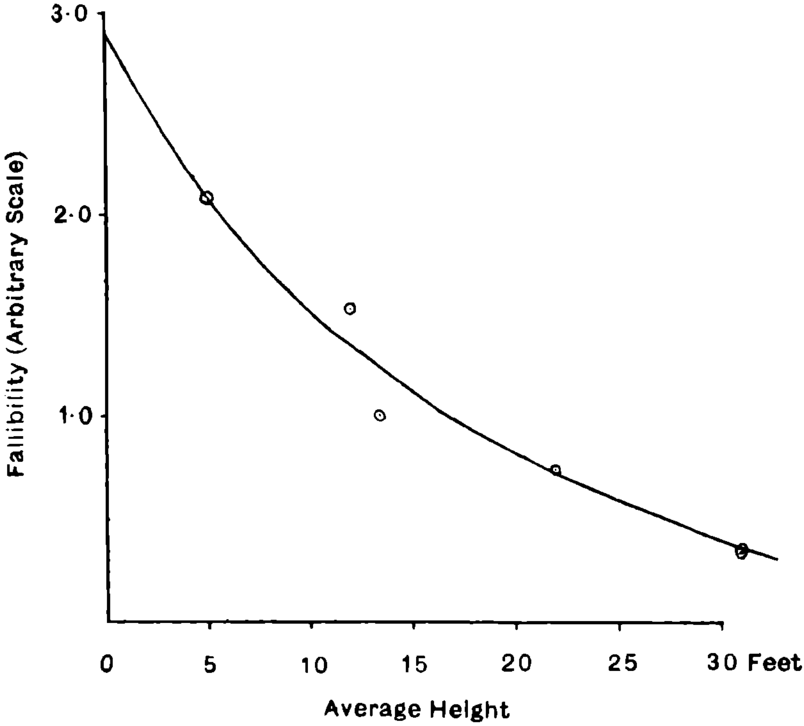
For determining A the following data are available. The other rays are visibly unsymmetrical.

Balawali ray AX.	Length a 4650′ h 5′	Error	·099′ 1st day.
			·073′ 2nd day.
			·037′ 3rd day.
Balawali ray CZ.	Length a 5940′ h 22′	Error	·024′ 1st day.
			·005′ 2nd day.
			·033′ 3rd day.

The resulting values of A are $4 \cdot 3, 3 \cdot 2, 1 \cdot 6, 2 \cdot 0, 0 \cdot 4$ and $2 \cdot 7 \times 10^{-7}$ respectively.

Mean value of $A = 2 \cdot 4 \times 10^{-7}$

Fig. 9



The smooth curve is Fallibility $= 2{\cdot}9e^{-{\cdot}065\,h}$.

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This expression is for the probable error of one day's work, observations being made at several hours. Increasing the number of days does not decrease this probable error, because it consists largely of unsuspected systematic error.

Consider next the visible asymmetry. Take a ray a feet long with b feet of sand on one side and none on the other. Let the temperature gradient over the sand be $\delta_1°$ F per 1000 feet, and that over the water be $\delta_2°$ F per 1000 feet. δ_1 will generally be negative and δ_2 positive.

Proceeding as in the example previously worked out, we find

$$\text{Error} = 2\cdot 6 \times 10^{-10}\, b\,(a-b)\,(\delta_2-\delta_1) \text{ feet}$$

$$= 2\cdot 6 \times 10^{-10}\, b.c.\,(\delta_2-\delta_1) \text{ feet, where } c \text{ is the distance over water.}$$

The same formula may be used when there is some sand on both sides, but b feet more on one side than on the other. It will seldom be practicable to measure $\delta_2-\delta_1$, but Balawali rays BY and DV are typical, and a value can be deduced from them. $\delta_2-\delta_1$ depends of course on the height. As before it will be assumed proportional to $e^{-\cdot 065h}$.

The data are

Balawali ray BY. a 4980 . b 1700′. h 14′. c 3280′. Mean error ·090′.
Balawali ray DV. a 3300′. b 1850′. h 12′. c 1450′. Mean error ·084′.

These give $\delta_2-\delta_1 = 150°$ F and 260° F respectively for $h = 0$. Mean 205° F, which is equivalent to 105° F per 1000′ at 10 feet. This is not inconsistent with the results of direct measurement. It is applicable to observations made during the day while the sun is shining. A probable error must be assigned to this estimate of the systematic error. 33 % is a reasonable value, since a departure of 100 % would give results which may be considered possible but improbable.

The final expression for the total error is then

$$\left\{2\cdot 6 \times 10^{-10}\, b.c.\, e^{-\cdot 065h}\, 205\right\}\left\{1 \pm \tfrac{1}{3}\right\} \pm 2\cdot 4 .\, 10^{-7} a^{\frac{3}{2}} e^{-\cdot 065h} \text{feet.}$$

Combining the two probable errors gives approximately

$$\text{Error} = e^{-\cdot 065h}\, 10^{-8}\left[5\, b.c. \pm 1\cdot 7a^2 \sqrt{\left(\frac{b}{a}\right)^2\left(\frac{c}{a}\right)^2 + \frac{200}{a}}\right] \text{feet}$$

a, b, c, and h in feet. The sign of the systematic error is such that the height of the sandy side above the other appears to be too great.

The limits of length between which the formula is intended to apply are from ½ mile to 2 miles: and of height from 4 feet to 35 feet.

The precise form of this statement gives it an appearance of exactitude which it in no way possesses. The systematic errors given by it should not generally be applied as a correction to observations. It is only intended to serve as a guide when considering the effects of length, height, and asymmetry on the probable accuracy of alternative sites. If, as is suggested later, important crossings are made at several sites, the sites should be so chosen that the indicated systematic errors, if any, should be of opposite sign in different rays.

Table 7 shows the result of applying this formula to the Balawali rays. It will be seen that the systematic errors are of the correct order of magnitude, and that the differences between observed and calculated errors are fairly well represented by the probable errors assigned. The crossing at Damukdia has been included, the observed error being given on the assumption that the water gauge gave the correct result.

TABLE 7

Crossing	Length feet	Distance over water	Excess of sand feet	Height feet	Observed Errors feet	Mean observed Error feet	Calculated Error feet
Balawali AX	4650	4650	0	5	+ ·099 + ·073 + ·037	+ ·07	·00 ± ·05
BY	4980	3280	1700	14	+ ·061 + ·108 + ·100	+ ·09	+ ·11 ± ·05
CZ	5940	2840	700	22	+ ·024 + ·005 + ·033	+ ·02	+ ·02 ± ·03
DV	3300	1450	1850	12	+ ·081 + ·084	+ ·08	+ ·06 ± ·03
EW	4170	2320	1850	22	+ ·023 + ·059	+ ·04	+ ·05 ± ·02
*EW	4170	1320	850	22			+ ·03 ± ·02
Damukdia	6150	4250	1900	22	... ...	+ ·07	+ ·10 ± ·04

* Ray EW has been computed on two assumptions. (1) That standing crops resemble water. And (2) That they resemble sand. Neither assumption is definitely contradicted by the observations.

The importance of symmetry is very great, as its absence destroys the chance of improved accuracy from repeated observations. At first sight it is best obtained by putting the ray entirely over water. Symmetry, of course, not only implies similar surfaces beneath the ray, but also equality of height above that surface. This can seldom be fulfilled unless the level be set up at the water's edge with the ray passing within 5 feet of the surface. Height may often be gained by going back from the water over similar distances of similar surfaces, but this lengthens the ray, and also involves the risk that surfaces which appear fairly similar may in fact have very different radiating powers.

Table 8 gives some typical values of the probable errors when $b = 0$, i.e. when there is an equal quantity of sand on either side. Both sides must have the same kinds of surface and be of similar heights.

TABLE 8. *Probable errors in feet*

Total length feet \ Average height feet	5	10	20	30
2000	·013	·009	†	†
2500	·018	·013	†	†
3000	·024	·017	·009	†
4000	·036	·026	·014	·007
5000	·051	·037	·019	·010
7500	·094	·067	·035	·018
10000	·144	·104	·054	·028

† The formula gives a p.e. of less than ·01. Other sources of error will probably vitiate this result.

Tables 9, 10 and 11 give some typical values for unsymmetrical sites, in which there is no sand on one side and *b* feet on the other, such as Balawali ray BY. *c* is the width of water. The total length $=c+b$.

TABLE 9. $c=2500$ *feet.*

$b=$ Excess of sand \ Average height feet	5	10	20	30
0	00 ± ·02	00 ± ·02	00 ± ·01	00 ± *
500	·05 ± ·03	·03 ± ·02	·02 ± ·01	·01 ± ·01
1000	·09 ± ·05	·06 ± ·03	·03 ± ·02	·02 ± ·01
2000	·18 ± ·08	·13 ± ·06	·07 ± ·03	·03 ± ·02

* See note to Table 8.

TABLE 10. $c=5000$ *feet.*

$b=$ Excess of sand \ Average height feet	5	10	20	30
0	00± ·06	00± ·04	00± ·02	00± 01
500	·09± ·08	·06± 06	·03± ·03	·02± ·02
1000	·18± ·10	·13± ·07	·07± ·04	·03± ·02
2000	·36± ·16	·26± ·12	·13± ·06	·07± ·03

TABLE 11. $c=7500$ *feet.*

$b=$ Excess of sand \ Average height feet	5	10	20	30
0	00± ·11	00± ·08	00± ·04	00± ·02
500	·13± ·13	·10± ·10	·05± ·05	·03± ·03
1000	·27± ·17	·19± 12	·10± ·06	·05± ·03
2000	·54± ·24	·39± ·17	·20± ·09	·10± 05

Water-gauges.—The water-gauge is independent of refraction, but its use involves other difficulties, notably wind and current. There are, however, generally hours of the day during which the air is absolutely still, and in a slow moving river a suitable choice of site should bring the effects of the current down to small limits. Little confidence might perhaps be placed in water gauge observations at a single site, but if accordant results were obtained from different sites some miles apart, some sites erring from perfection in one direction and some in another, the results might be considered reliable.

Three methods were used at Damukdia (Prof. Paper 7) with the following results.

Method	*Height difference*
Vertical Angles	2·139 feet.
Target	2·132 „
Water-gauge	2·212 „

From the point of view of refraction the crossing was badly unsymmetrical (Fig. 6). The first two methods are both burdened with similar error due to this cause, and their agreement has no significance. Table 7 indicates that the water-gauge has given the best result.

The following discussion of the effects of the current is not made in the hope of being able to apply an accurate correction to the results obtained at an imperfect site, but of indicating the magnitude and direction of the errors which may occur.

(1). The crossing must be made at right angles to the current. More precisely, a line passing through one gauge and crossing the river orthogonally to the current at every point, must pass close to the other gauge. If the fall of the river be 6 inches per mile, and if it is hoped to avoid an error of more than ·02 feet (a high standard of accuracy), the gauges must be correctly placed to within 200′. The worst source of error would be a shoal in midstream, with the water flowing transversely across it. Any site in which the water shallows in the centre should be avoided. This condition was not satisfied in the crossing of the Indus at Dera Ismail Khān in 1906, referred to in G.T.S. Vol. XIX. Appendix V. A fortnight after the crossing had been made, the river, having fallen, had divided into several separate channels.

(2). The lines of flow must be straight to avoid piling up on the outer side. Consider a bend of the river in which the average radius of curvature is R miles. Considering the horizontal forces on small cube of side δs, moving with radius of curvature r, we have

$$\delta p\,(\delta s)^2 = \frac{v^2}{r}\,m\,(\delta s)^3,$$ where v is the velocity, m is the mass of unit volume and p is the pressure.

Also $\delta p = m\,g\,\delta h$, where h is the height of the water surface.

$$\therefore\ \delta h = \frac{v^2}{gr}\,\delta s$$

As an approximation replace v and r by their average values. Let H be difference in height between the inner and outer margins of the river.

Then $H = \dfrac{v^2}{gr} \times$ (width of river)

For example, if $v = 3$ feet per second, and $r = 4$ miles, and the width = 1 mile, H = ·070 feet.

On large, slow moving rivers there should be no difficulty in find ing sites more favourable than this.

(3). The velocity of the water passing the two gauges must be equal. In elementary hydrodynamics the loss of head associated with a velocity of v is $\frac{v^2}{2g}$. It may be doubted whether such elementary considerations can be applied to the water of a river in which friction may be a more dominant factor than is the conservation of free energy, but it is unwise to run any risk. If the velocity at one gauge be v_1, and at the other v_2, the error will be $\frac{1}{2g}(v_1^2 - v_2^2)$. If $v_1 = 3$ feet per second and $v_2 = 1$, the error will apparently be ·125 feet, a serious matter. A safe rule would be to erect the gauges in water with a velocity of less than 1 foot per second.

The following is a summary of the requirements of a good site for a water-gauge crossing.

(1) There must be no central shallows.

(2) The velocity must be small. If the current anywhere exceeds 5 feet per second, water-gauges are probably not worth trying. Errors are likely to increase as the square of the velocity.

(3) The current must be straight. A narrow river may be allowed a smaller radius of curvature than a wide one.

(4) The air must be still.

(5) A narrow crossing is of course desirable, but not at the expense of a high velocity. Errors are only likely to increase as the first power of the width.

(6) The correction for the Earth's rotation (See Appendix G) must not be unduly large.

The reasons given in G.T.S. Vol. XIX Appendix V for preferring "Vertical Angles" to Water-gauges seem inconclusive. Of the three examples quoted there, the Damukdia result is favourable to the water-gauge. The Indus crossing was a bad site, as mentioned above. On the Chenāb the current seems to have been fast, since pools had to be dug in the sand to get an unagitated water surface. For such a narrow crossing (12 chains) the ordinary method of crossing would of course always be preferable. See also G.T.S. Vol. XIX pages 83 and 84.

No experiments were made at Balawali as the river was too low, and long crossings could only be got along the length of the river.

The Micrometer eye-piece levels.—The two levels which have been fitted are American Binocular Nos. 2698 and 6728. At infinite focus the values of a micrometer division average 1·26″ and 1·30″ respectively. (See Appendix C). The general arrangements for simultaneous reciprocal observations are as follows.

The levels are set up on either side of the river at approximately equal heights above the water, and equidistant from terminal bench-marks on either bank (About 3 chains). Making these back rays equal eliminates the necessity for applying any correction for earth's curvature to them. Observations are made to 9-inch targets, painted diagonally black and white, fitted round the object glasses of the two levels. A staff is put up on each terminal B.M. It is of course necessary that these staves should be the same ones as are placed on the bench-marks when connecting them to the level lines, or else that suitable corrections for zero error be applied.

On each bank collimating staves are set up at half a chain and $2\frac{1}{2}$ chains respectively from the station. An auxiliary collimating station is placed half way between these two staves. Collimating is done by the usual two-staff method, except that the horizontal wire being a moveable one, there is no need to adjust the spirit level or diaphragm, the result of collimating being to give the micrometer reading which brings the horizontal wire to the position of no collimation. A fixed wire, out of the way in the bottom of the field of view, serves as a zero from which to measure whole turns of the micrometer head.

The procedure at a station is :—

(1) Collimate, finishing on the station which is unequally placed between the staves. The readings must be properly recorded on a form provided for the purpose.
(2) Set the micrometer to the last recorded position of no collimation error, and read the back staff.
(3) Put up signal flag.
(4) When both observers' signal flags are up, commence making intersections on the target on the opposite level. When 10 are complete strike the flag, but continue observing until both flags are down.
(5) Repeat every hour, including the collimation observation.

Then the difference between the micrometer reading of the line of no collimation, and the mean reading of the target (converted into arc), is the elevation or depression. Heights of instrument are given by the back staff readings, and the difference in the heights of the bench-marks is computed as for ordinary reciprocal vertical angles observed with a theodolite. Better uniformity of results will generally be obtained by using the daily mean value of the collimation, instead of the hourly values.

No advantage comes from making a great number of observations. Except when the air is very unsteady, a single intersection of a good mark is as reliable as a single determination of collimation, or as the equality of the angles of refraction. But as micrometer intersections are so easy to make, it seems desirable to make about ten at a time, and so to eliminate any unnecessary error caused by bad intersection. Nor is there much profit in observing more than once an hour. It is also not essential that the observations be simultaneous to within less than a few minutes; but exact coincidence is easy to obtain.

Collimation is carried out as follows, the observations being recorded as in Fig. 10.

(1) Set up half way between the staves.
(2) Set the micrometer to any convenient reading near the position of no collimation. This setting should be varied each time, keeping if possible within ten divisions of the true position.

Fig. 10

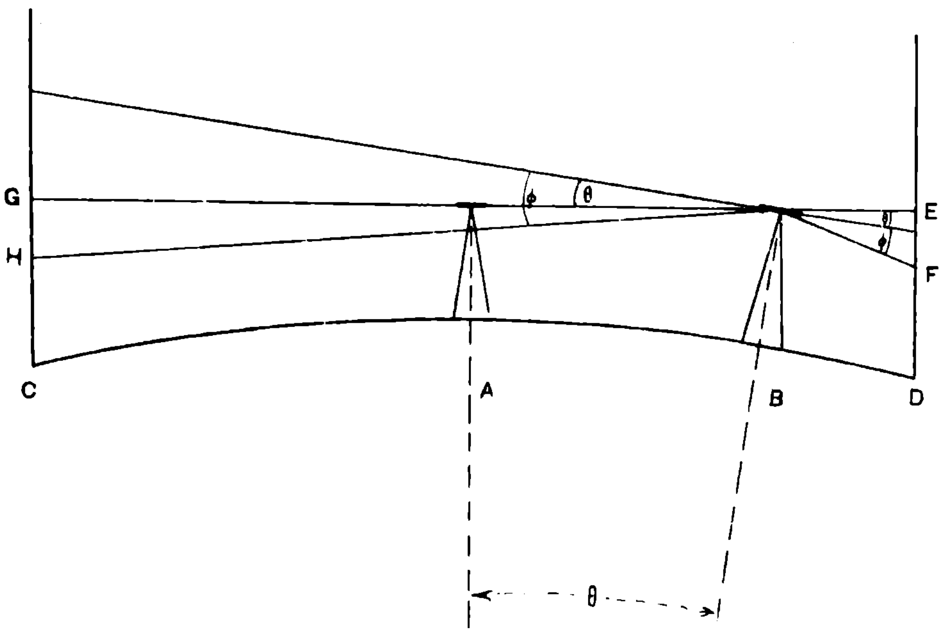
COLLIMATION

Date 17.3.27 Instrument 6728 Observer B.L.G.

Distance between Staves 3 Chains

Time and Setting	Near	Far		Far—Near
09. 18	4·686	4·019		- 0·667
990	4·023	3·364		- 0·659
		- 0·010	5/4 Diff	- 0.008
Check →	4·021	3.354	← Sum	
		9800	← Micro	
		980·7	← Corrected	

Fig. 11



(3) Read the staves. Under "Near" record the reading of the staff towards which the level is afterwards going to be moved.

(4) Set up again so that the distances of the staves are as 5 to 1. Read the near staff first and then the far. Up to this point the micrometer must not have been turned.

(5). On the form add 5/4 of the change of height difference to the last reading on the far staff. Intersect this reading by turning the micrometer head, and record the micrometer reading. This is the position of no collimation. It should occasionally be checked by reading the near staff with the new setting, and seeing that the height difference then arrived at agrees with that found at equal distances. This does not give increased accuracy, but is a check against the correction having been applied with the wrong sign. It is important that the staves should remain truly vertical. As they are standing all day, this is apt to be overlooked, and may cause a systematic error.

Collimation is the most unsatisfactory part of the observation. A single measure cannot be trusted to be correct within less than 3 or 4 seconds. Frequent repetition of the process is necessary, and this is very troublesome. There is also the possibility of systematic change of collimation with change of focus, (although, if truly systematic, this is overcome by interchanging instruments) and of error due to the fact that the two-staff method involves rays of unequal length, and so of unequal refraction. It seems unlikely that the mean of any reasonable number of measures of collimation error at a single place can be trusted to be correct within less than 1 second. As this is only .03 feet at a distance of one mile, it is not extremely serious.

It is obviously desirable to interchange levels from one side of the river to the other every day, and this should be done, but it is no complete remedy, as collimation is not sufficiently constant. (See Appendix B). Certain accuracy can be obtained by Gauss' method, but at the expense of simplicity. A micrometer eye-piece might be fitted to a reversible level, but none of the old reversible levels has the accuracy or convenience of the American Binocular. The large pattern Zeiss level is in many ways suitable, but it cannot immediately be fitted. The existing cross wires are on the inside faces of the two object glasses, and the range of focus is insufficient to bring the image of the target to a position where a moving wire can conveniently be placed. This can probably be overcome by fixing an auxiliary lens over the existing object glass. The only requirement with regard to the fixing of this lens and of the eye-piece, is that they should not move with reference to the rest of the telescope, when the telescope is rotated through 180° about its longitudinal axis. It is not essential that they should always return to the same position after being interchanged.

The procedure with such a level would be as follows. The eye-piece must always be put on in such a position that increasing micrometer readings correspond to increasing elevations when the bubble is (say) on the right of the telescope. This can be ensured mechanically. No separate measures of the position of no collimation are required, but the reciprocal observations should be made in all four positions. Three intersections with the micrometer would suffice in each position: total 12.

$$\text{Then Elevation} = \frac{(\text{Sum of all bubble right readings}) - (\text{Sum of all bubble left readings})}{12}$$

converted into arc.

For reading the back staff the position of no collimation is required. For any of the four positions this may be obtained by immediately computing the elevation as above (expressed in micrometer divisions) and subtracting it from the mean micrometer reading of the target when intersected in the required position. The reading of the back staff must therefore be postponed until after the target has been read.

Other means of utilising the Zeiss level are under consideration. If they mature, the observation will differ in detail from the above, but will be similar in principle.

If a site can be found in the middle of the river, it suffices to set up one level there, with equidistant staves on either bank. Collimation error need not then be accurately determined, and the assumption that the two angles of refraction at the centre station are equal, is at least as good as the usual assumption of their equality at either bank. Such a a ray will necessarily be very close to the water.

Recommendations for future work.—Two classes of work require separate consideration.

(1) Lines of the primary level net, in which an error of ·100 foot is extremely undesirable.

(2) Work of lesser accuracy in which an occasional error of this size is not serious.

For the latter, crossing by simultaneous observations with American Binocular levels (or Zeiss if they can be fitted satisfactorily) fitted with micrometer eye-pieces is recommended. The procedure to be as described above, and observations to be made on one, or at most two days only. Observations should be carried on throughout the day. There is no special advantage in the normal times of minimum refraction. A strong wind or clouds are favourable, being inconducive to the forming of large temperature gradients. As a precaution against gross error in determining the collimation, or of systematic change of collimation with change of focus, it is essential that the levels change sides at least once. If observations last two days, this is not much trouble. If they last one day only it will waste a couple of hours in the middle of the day. Nevertheless it should be done.

The equipment required, in addition to that generally carried by two levellers is :—

(1) The two levels and targets.

(2) Two extra staves. (Unless Zeiss levels are used).

(3) A plane-table with sight rule, etc., with which to make a rough measure of the distance.

(4) Two flags for signalling.

Observatory tents and concrete bench-marks are not necessary, and one day should suffice for all preparations.

Unless it is possible for the Officer in charge of the party to select the site himself, it is suggested as a working rule, that for crossings between half a mile and 1 mile wide the stations should be put at the water's edge, unless at least one foot of height is gained for every chain by which the further station is distant from the water. (See Tables 8, 9, 10, 11) For narrower rivers a greater increase of height per chain should be required; for very wide rivers, less.

The difference between the heights of the stations should not exceed two feet, and if close to the water it should be less than six inches.

For short crossings (say under half a mile), in work of low accuracy it may be possible for one observer to make the crossing. For rays within 10′ of the water the coefficient of refraction should be taken as 0·3, and for higher rays 0·1. The probable error of this assumed coefficient may be taken to be 0·2 in the first case, and 0·1 in the second. The resulting probable errors of the difference of height are

$l^2 \times 0 \cdot 3$ feet for rays within 10 feet of the water,

$l^2 \times 0 \cdot 15$ feet for rays more than 10 feet above the water,

where l is the length in miles.

It is of course possible for errors of three times this amount to occur.

For the primary net.—There are not many unbridged river crossings in the primary net, and it will be proper to go to some expense to cross them; it is suggested that some or all of the following recommendations be acted on.

(1) That levels with micrometer eye-pieces be used. Procedure as above. The levels to change sides daily. If it is not practicable to change sides, and if an irreversible instrument is used, collimation must be done by Gauss' method, on account of the risk of change of collimation with change of focus.

(2) That crossings be made at a number of sites covering some miles of the banks. No elaborate preparations should be made at each site. One day's work should be done at each, and not more than one day should be wasted between each.

(3) If suitable sites can be found, and if high level crossings are not available, the level may be set up in the middle of the river as referred to above. A very firm base is not required. It is only necessary that the level should be steady while the observer is sitting still, and that it should not change height by more than half an inch while he changes from one side of it to the other.

By using three levels in such a case an increase of accuracy can be obtained, each half of the river being crossed by reciprocal angles as in Fig. 8. The centre observer works with each of the other two in turn. It is not necessary to measure the height of his instrument on a back staff, and if the rays are equal his collimation error is immaterial.

If the crossing is more than $1\frac{1}{2}$ miles wide every endeavour should be made to do this, even to the extent of sinking a wooden frame full of stones in shallow water. Such a crossing is twice as good as a simple crossing at water level.

(4) Sites just below river junctions should be avoided, as a precaution against systematic differences of water temperature on either side.

(5) The whole of the observations should not be made when a prevalent wind is blowing from one side to the other. The observations of any one day may be so done, but not the whole of the work.

(6) Increased height may perhaps be obtained artificially by staging, as is done in triangulation. But it is probable that the necessary preparations would lead to an expenditure of time, which would have been better spent on making more numerous crossings.

(7) Water-gauges should also be used in several well separated sites if possible; if the conditions are suitable it is possible that they would give more consistent results than the levels. The observations might perhaps be made on the same days as the other crossings, so avoiding waste of time.

APPENDIX A

A SOURCE OF ERROR IN THE TWO-STAFF METHOD OF COLLIMATING

See Fig. 11 facing page 14. This is caused by the fact that when the level is set up in the unequal distance position, earth's curvature is not cancelled. There is additional indeterminate error due to refraction being unequal.

Suppose the zeros of the two staves to be at equal heights, and suppose the trial position of the horizontal wire to be correct. The height difference recorded when the level is set up at A will be zero. That recorded at B will not be zero, in spite of the absence of collimation error, because at B the bubble does not lie parallel to its position at A.

Agreement can only be got by introducing a collimation error ϕ, such that EF = GH, i.e. $\phi + \theta = \frac{CB}{BD}(\phi - \theta)$, where θ is the angle subtended by AB at the earth's centre.

i.e. $\phi = \frac{CB + BD}{CB - BD}\theta = \frac{CD}{2AB}\theta$ = half angular value of CD.

Example. If CD = 6 chains, ϕ = 2 seconds. Its direction is such that staff readings are too low.

For ordinary levelling this is of no consequence. For river crossings it may be neglected, provided it is the same in both instruments employed, i.e. provided they are both collimated with staves placed similar distances apart. The error due to unequal refraction will generally be much less. Except in the early morning it should seldom be more than one fifth as much.

APPENDIX B

THE CONSTANCY OF COLLIMATION ERROR IN LEVELS 2698 & 6728

See also Levelling Hand-book 1920, page 10.

Table 12 shows the measures made of collimation in the two levels between 3rd February 1927 and 18th March 1927. Doubtful values have been excluded, but any single one may be 3″ or 4″ in error. When several measures have been made during one day, they have been at about hourly intervals. A change of 3″ or 4″ in the daily mean may be considered to be an actual change in the position of the line of collimation.

The sudden changes are probably due to movement of the object glass.

TABLE 12

Position of Line of Collimation

Seconds, from an arbitrary zero

Date and Place	2698		6728		Remarks
	Single Measures	Mean	Single Measures	Mean	
3-2-27			− 1·7	0·0	
Dehra Dūn ...			+ 1·8		
			−11·6	−10·4	
			− 9·3		
4-2-27			−11·6	−11·6	
Dehra Dūn ...			−11·6		
5-2-27			− 3·9	− 3·8	
Dehra Dūn ...			− 3·7		
7-2-27			+ 3·0	+ 1·4	
Dehra Dūn ...			− 0·2		
1-3-27			− 9·1	−10·9	
Dehra Dūn ...			−12·8		
2-3-27			−18·1	−17·8	
Dehra Dūn ...			−17·4		
12-3-27	− 5·0	0·0	−24·9	−21·0	Railway journey
Balawali ..	+ 7·6		−17·4		
	− 2·2		−21·4		
	− 3·7		−16·2		
	+ 6·8		−24·9		
	− 2·7		−34·7	−33·8	Unexplained change in 6728
	− 0·8		−32·8		
13-3-27	+ 4·4	+ 2·7	−28·9	−29·5	
Balawali ...	+ 4·5		−32·1		
	+ 2·1		−30·3		
	+ 1·8		−30·6		
	+ 1·6		−28·9		
	+ 1·8		−29·5		
	+ 9·4	+ 8·2	−25·6		Unexplained change in 2698
	+ 8·5		−30·3		
	+ 3·0				
	+11·7				

Continued on next page.

TABLE 12 (*Continued*)

Position of Line of Collimation
Seconds, from an arbitrary zero

Date and Place	2698		6728		Remarks
	Single Measures	Mean	Single Measures	Mean	
14-3-27 Balawali ..	+10·4	+ 6·7	−25·6	−26·0	
	+ 9·0		−26·3		
	+ 7·8		−24·2		
	+ 5·6		−30·1		
	+ 7·9		−23·7		
	+ 1·6		−35·4	−33·6	Unexplained change in 6728
	+ 2·8		−35·4		
	+ 8·6		−30·1		
15-3-27 Balawali ...	+11·3	+ 7·0			
	+ 7·8				
	+ 5·4				
	+ 4·7				
	+ 8·5				
	+ 4·2				
17-3-27 Balawali ...	+10·7	+10·4	−57·4	−69·2	6728 changed as the result of a fall
	+ 8·1		−67·9		
	+12·3		−67·0		
	+ 1·6	0·0	−62·7		Unexplained change in 2698
	+ 2·8		−69·0		
	− 2·1		−78·9		
	− 1·7		−78·9		
	− 0·4		−71·9		
18-3-27 Balawali ...	+ 1·3	+ 2·8	−13·3	−15·3	6728 changed as the result of a fall
	− 0·4		−19·1		
	+ 2·1		−12·5		
	+ 3·7		−16·5		
	+ 1·8		−30·8	−30·2	Unexplained change in 6728
	+ 3·1		−27·6		
	+ 7·9		−28·9		
			−33·4		

APPENDIX C

AN IRREGULARITY IN THE MICROMETERS OF LEVELS 2698 & 6728

When the value of one division of the eye-piece micrometers of these levels was being determined, its value was found to be irregular. The irregularity is periodic, the period being one turn of the micrometer head. The amplitude (semi range) is 0·9″ in 2698, and 1·2″ in 6728. This is shown graphically in Figs. 12 and 13. In these figures the abscissæ are micrometer readings (one full turn = 100 divisions), and the ordinates are the difference between the actual corresponding angles of elevation, and those given by the best linear relationship. viz:— 1 division = 1·265″ for 2698, and 1 division = 1·305″ for 6728.

APPENDIX D

DETAILED RESULTS OF THE BALAWALI CROSSINGS

Tables 13, 14, 15, 16 and 17 give the following details:—

(1) The error. i.e. the error in height difference arising from the assumption that the angles of refraction at either end are equal.

(2) The coefficients of refraction deduced from the angles of refraction at either end. e.g. $\eta_A = \frac{100\Omega_A}{4650}$

(3) The average air temperature gradients deduced from these coefficients of refraction by the formula

$$\eta = 5\cdot4\left(\cdot0187 + \frac{dT}{dh}\right), \text{ or } \delta = \frac{1000\eta}{5\cdot4} - 19$$

where the gradient is δ°F per 1000 feet (N. B. δ positive indicates temperature increasing with height).

Inspection of the column "error" might at first sight make it appear that the normal time of minimum refraction is the most favourable for river crossings. But it so happened that a strong wind often sprang up at about 11 a. m. and the improvement is believed to be due to this.

Fig. 12

Level No. 2698

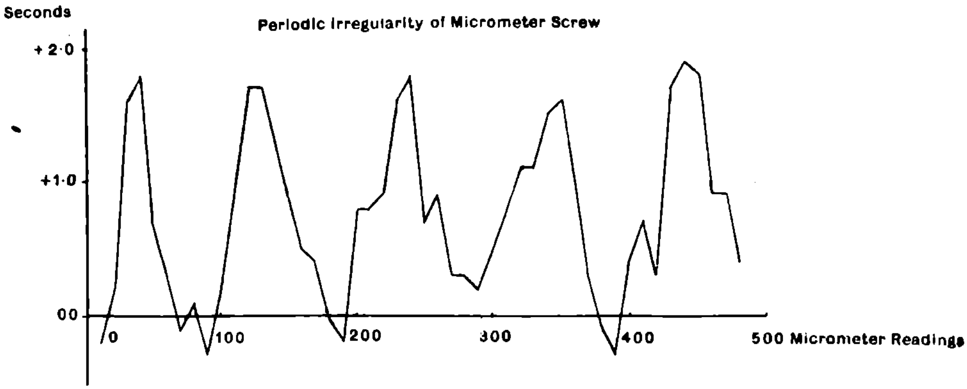
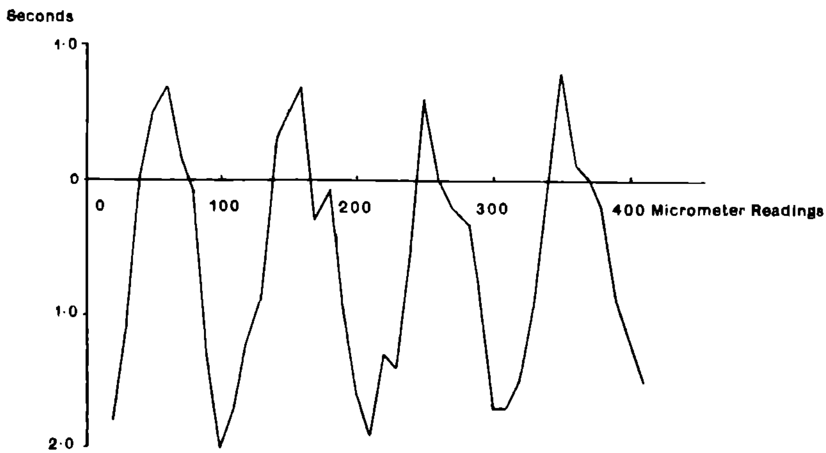


Fig. 13

Level No. 6728



Reg. No 14-1 M.D.D.1928.

Helio. S.I.O Dehra Dūn.

TABLE 13

Ray AX

Date and Time	Error feet	η_A	η_X	δ_A °F per 1000′	δ_X °F per 1000′	Remarks
12-3-27						
8 32	+ ·190	+ ·353	− ·009	+ 46	− 21	Still
10 01	+ ·084	+ ·233	+ ·073	+ 24	− 05	,,
10 54	+ ·077	+ ·253	+ ·107	+ 28	+ 01	Light wind
12 21	− ·018	+ ·218	+ ·252	+ 21	+ 28	,,
13 43	+ ·134	+ ·344	+ ·089	+ 45	− 03	,,
14 58	+ ·125	+ ·338	+ ·099	+ 44	− 01	,,
Mean	+ ·099					
13-3-27						
8 16	+ ·004	+ ·207	+ ·200	+ 19	+ 18	Very light wind
9 08	− ·003	+ ·264	+ ·270	+ 30	+ 31	,,
10 03	+ ·066	+ ·312	+ ·187	+ 39	+ 16	,,
10 55	− ·007	+ ·617	+ ·631	+ 95	+ 98	,,
11 48	+ ·169	+1·204	+ ·881	+ 204	+ 144	Light wind
13 09	+ ·045	+ ·815	+ ·730	+ 132	+ 116	,,
13 57	+ ·211	+ ·669	+ ·268	+ 105	+ 31	,,
14 56	+ ·095	+ ·510	+ ·329	+ 75	+ 42	,,
Mean	+ ·073					
14-3-27						
8 06	− ·088	− ·120	− ·048	− 41	− 28	Still
8 57	+ ·014	+ ·098	+ ·069	− 01	− 06	Light wind
9 45	+ ·075	+ ·294	+ ·150	+ 35	+ 09	,,
10 34	+ ·063	+ ·365	+ ·246	+ 49	+ 27	,,
11 21	+ ·068	+ ·503	+ ·373	+ 74	+ 50	Wind
12 14	+ ·077	+ ·554	+ ·409	+ 83	+ 57	Strong wind
13 38	− ·029	+ ·543	+ ·598	+ 81	+ 92	,,
14 49	+ ·068	+ ·650	+ ·520	+ 101	+ 77	,,
Mean	+ ·037					
Range of variation of δ_A and δ_X				245	172	

TABLE 14

Ray BY

Date and Time	Error feet	η_B	η_Y	δ_B °F per 1000′	δ_Y °F per 1000′	Remarks
12-3-27						
9 23	...	...	...	...	...	...
10 26	+ ·114	+ ·258	+ ·069	+ 29	− 06	Still
11 51	+ ·051	+ ·259	+ ·175	+ 29	+ 13	Light wind
12 52	+ ·085	+ ·224	+ ·083	+ 21	− 04	"
14 24	+ ·071	+ ·218	+ ·101	+ 21	00	"
15 28	− ·015	+ ·182	+ ·207	+ 15	+ 19	"
Mean	+ ·061					
13-3-27						
8 41	− ·035	+ ·170	+ ·228	+ 12	+ 23	Very light wind
9 36	+ ·090	+ ·320	+ ·171	+ 40	+ 13	"
10 30	+ ·089	+ ·319	+ ·170	+ 40	+ 12	"
11 19	+ ·070	+ ·260	+ ·143	+ 29	+ 07	Light wind
12 17	+ ·132	+ ·347	+ ·128	+ 45	+ 05	"
13 35	+ ·161	+ ·322	+ ·055	+ 41	+ 09	"
14 28	+ ·178	+ ·212	− ·084	+ 20	− 35	"
15 21	+ ·182	+ ·179	− ·124	+ 14	− 42	"
Mean	+ ·108					
14-3-27						
8 33	+ ·102	+ ·197	+ ·027	+ 17	− 14	Still
9 19	+ ·117	+ ·166	− ·028	+ 12	− 24	Light wind
10 07	+ ·139	+ ·130	− ·101	+ 05	− 38	"
11 00	+ ·153	+ ·116	− ·139	+ 02	− 45	"
11 47	+ ·184	+ ·081	− ·224	− 04	− 60	Wind
12 41	+ ·012	+ ·026	+ ·005	− 14	− 18	Strong wind
14 15	+ ·008	+ ·034	+ ·020	− 13	− 16	"
15 18	+ ·082	+ ·056	− ·080	− 09	− 34	"
Mean	+ ·100					
15-3-27						
10 49		+ ·165		+ 15		
12 01		+ ·047		− 10		
13 06		+ ·051		− 10		
14 05		+ ·056		− 09		
14 56		+ ·066		− 07		
Range of variation of δ_B and δ_Y				59	83	

TABLE 15

Ray CZ

Date and Time	Error feet	η_C	η_Z	δ_C °F per 1000′	δ_Z °F per 1000′	Remarks
12-3-27						
9 41	+ ·036	+ ·140	+ ·098	+ 07	− 01	Still
10 37	+ ·052	+ ·129	+ ·068	+ 05	− 06	"
12 01	− ·031	+ ·044	+ ·081	− 11	− 04	Light wind
13 03	− ·028	+ ·044	+ ·078	− 11	− 05	"
14 44	+ ·030	+ ·027	− ·008	− 14	− 20	"
15 40	+ ·088	+ ·076	− ·026	− 05	− 24	"
Mean	+ ·024					
13-3-27						
8 54	− ·115	+ ·129	+ ·264	+ 05	+ 30	Very light wind
9 48	+ ·003	+ ·178	+ ·174	+ 14	+ 13	"
10 43	+ ·028	+ ·200	+ ·166	+ 18	+ 12	"
11 32	− ·082	+ ·062	+ ·158	− 08	+ 10	Light wind
12 28	− ·050	+ ·070	+ ·128	− 06	+ 05	"
13 46	+ ·031	+ ·118	+ ·082	+ 03	− 04	"
14 41	+ ·158	+ ·145	− ·039	+ 08	− 12	"
15 32	+ ·065	+ ·054	− ·022	− 09	− 15	"
Mean	+ ·005					
14-3-27						
8 45	+ ·133	+ ·146	− ·009	+ 08	− 21	Still
9 29	+ ·055	+ ·033	− ·032	− 13	− 25	Light wind
10 20	− ·028	+ ·008	+ ·040	− 18	− 12	"
11 07	+ ·075	− ·017	− ·105	− 16	− 38	Wind
11 57	+ ·065	− ·028	− ·104	− 14	− 38	Strong wind
12 52	− ·188	− ·041	+ ·179	− 11	+ 14	"
14 36	+ ·023	+ ·022	− ·004	− 15	− 20	"
15 29	+ ·127	+ ·035	− ·113	− 13	− 40	"
Mean	+ ·033					
Range of variation of δ_C and δ_Z				36	70	

TABLE 16

Ray DV

Date and Time	Error feet	η_D	η_V	δ_D °F per 1000′	δ_V °F per 1000′	Remarks
17-3-27						
8 16	+ ·114	+ ·348	− ·081	+ 46	− 34	Still
9 12	+ ·040	− ·004	− ·157	− 20	− 48	"
10 13	+ ·086	+ ·021	− ·303	− 15	− 75	"
11 11	+ ·151	+ ·199	− ·374	+ 18	− 88	Light wind
12 08	+ ·091	+ ·119	− ·225	+ 03	− 60	Wind
13 12	+ ·073	+ ·055	− ·219	− 09	− 59	Light wind
14 11	+ ·013	− ·185	− ·236	− 54	− 63	Wind
15 15	...	− ·189	...	...	− 54	Strong wind
Mean	+ ·081					
18-3-27						
8 16	+ ·022	+ ·248	+ ·165	+ 27	+ 12	Light wind
9 15	+ ·135	+ ·072	− ·439	− 06	− 100	"
10 11	+ ·127	+ ·032	− ·448	− 13	− 102	"
11 11	+ ·087	− ·053	− ·380	− 29	− 89	Wind
12 13	+ ·097	− ·013	− ·380	− 21	− 89	Strong wind
13 14	+ ·062	+ ·125	− ·110	+ 04	− 39	"
14 20	+ ·079	+ ·049	− ·250	− 10	− 65	"
15 14	+ ·065	+ ·002	− ·244	− 19	− 64	"
Mean	+ ·084					
Range of variation of δ_D and δ_V				100	114	

TABLE 17

Ray EW

Date and Time	Error feet	η_E	η_W	δ_E °F per 1000′	δ_W °F per 1000′	Remarks
17-3-27						
8 01	+ ·002	+ ·152	+ ·146	+ 09	+ 08	Still
9 00	+ ·061	+ ·070	− ·075	− 06	− 33	"
10 00	+ ·032	− ·008	− ·084	− 20	− 35	"
10 59	+ ·045	+ ·037	− ·071	− 12	− 32	Light wind
12 00	+ ·110	+ ·242	− ·020	+ 26	− 23	Wind
12 59	− ·051	− ·121	− ·001	− 41	− 19	Light wind
13 58	− ·017	− ·088	− ·046	− 35	− 28	Wind
14 56	− ·001	− ·046	− ·044	− 28	− 27	Strong wind
Mean	+ ·023					

(Continued)

TABLE 17 (*continued*)

Ray EW

Date and Time	Error feet	η_E	η_W	δ_E °F per 1000′	δ_W °F per 1000′	Remarks
18-3-27						
8 01	+ ·055	+ ·307	+ ·175	+ 76	+ 13	Light wind
9 00	+ ·148	+ ·162	− ·188	+ 49	− 54	,,
9 59	+ ·089	+ ·078	− ·131	− 05	− 43	,,
11 00	+ ·112	+ ·104	− ·162	00	− 49	Wind
11 57	+ ·082	− ·008	− ·203	− 20	− 57	Strong wind
13 00	− ·022	+ ·027	+ ·081	− 14	− 04	,,
14 00	− ·007	+ ·120	+ ·136	+ 03	+ 06	,,
15 01	+ ·014	+ ·127	+ ·092	+ 04	− 02	,,
Mean	+ ·059					
Range of variation of δ_E and δ_W				117	70	

APPENDIX E

DIRECT MEASURES OF THE AIR TEMPERATURE GRADIENT

In 1914 an apparatus was designed by the National Physical Laboratory at the request of Dr. J. de Graaff Hunter, Sc.D. for the measurement of air temperature at different heights, in order to make a direct determination of the gradient.

The temperatures are measured by changes in the resistances of a number of platinum wires suspended at the desired heights. An equal current is passed through them all, and also through a standard coil of manganin, whose resistance is invariable with temperature. The resistance of each platinum wire is measured by comparing the drop of potential in it, with the drop in the manganin. The thermometers are hung from a 100 foot portable mast, as used for triangulation or wireless aerials. They are screened from the sun and from radiation from the ground.

This apparatus has been set up at Dehra Dūn and at Balawali. Tables 18, 19 and 20 give a summary of the results. It was also hung from the bridge at Balawali, in the expectation that the gradient measured by it would show some agreement with that deduced from refractions simultaneously observed at Station B. They failed to do so, possibly owing to the influence of the mass of brick and metal forming the bridge, but more probably on account of observations at one place not being a fair sample of the ray as a whole. They confirm the existence of a reverse gradient over water during the day, ranging between 240° and 40° F per 1000 feet at 7½ feet from the surface.

TABLE 18

Dehra Dūn. Between 11.00 and 16.00 hours.

Time	Average gradients in °F per 1000 feet at heights of				
	13 feet	31 feet	48 feet	65 feet	83 feet
11 36	− 90	+ 11	+ 30	− 17	− 42
13 02	−101	− 15	− 34	+ 45	− 48
14 43	− 70	− 36	− 23	− 03	− 12
11 15	− 67	− 29	− 08	− 11	− 20
12 00	−134	+ 05	− 38	+ 03	+ 15
12 39	−105	+ 18	− 38	+ 21	+ 17
13 20	−154	− 13	− 17	+ 32	− 47
14 04	− 57	− 41	00	− 30	− 05
14 32	−102	− 33	+ 06	− 26	− 09
15 15	− 87	− 22	− 22	− 31	− 16
11 05	− 09	− 04	− 13	− 19	− 09
11 44	− 19	− 07	+ 12	− 22	+ 08
12 32	− 24	− 37	+ 29	− 33	+ 18
Mean ...	− 78	− 16	− 09	− 07	− 12
Standard Deviation ...	44	20	23	26	23

Each entry is the mean of a group of 6 or 8 measures.

The Standard Deviation has been computed for the purpose explained on page 6.

Table 19 gives the mean results at Balawali camp in °F per 1000 feet. The reversal at 48 feet is notable. This is believed to be due to a layer of cold air blowing off the river.

TABLE 19. Balawali camp

Height	13 feet	31 feet	48 feet	65 feet	83 feet
Between 8.00 & 11·00 hours Mean gradient ...	−117	−15	+07	−17	−14
Standard Deviation ...	45	29	20	18	27
Between 11·00 & 16·00 hours Mean gradient	−124	−31	+02	−15	−23
Standard Deviation ...	48	26	27	12	18

Table 20 gives the results at Balawali bridge. The two blanks are caused by one thermometer being in the sun and the adjacent one being in the shade, vitiating the measure of the gradient between them.

TABLE 20

Time	Gradient in °F per 1000 feet at heights of		
	7½ feet	16½ feet	25½ feet
h m			
10 03	+146	−18	−95
11 01	+243	+05	−38
12 00	+ 59	+14	+18
13 01	+ 56	00	+14
14 04	...	−02	+34
15 01	+ 41	...	−04
Mean	+109	00	−12

N.B.—A positive gradient indicates temperature increasing with height.

The gradients at 16 feet deduced from observation of vertical angles at Station B were.

h m	
10 49	+15 °F per 1000 feet
12 01	−10 ,,
13 06	−10 ,,
14 05	−09 ,,
14 56	−07 ,,
Mean	−04 ,,

They are of the correct order of magnitude, but show no more detailed resemblance. The readings at 7½ feet show general agreement with those deduced from ray AX. See Table 13, particularly for 13th March 1927.

APPENDIX F

REFRACTION

Given :—(1) The law of refraction $\mu \sin \phi$ = constant,
where μ is the refractive index and ϕ the angle between the ray and the normal to the surface.

(2) The relation between the density of air and its refractive index, viz $\mu - 1 = k\rho$,
where ρ is the density, and k is a constant

(3) The " Perfect Gas " law. $P = c\,T\,\rho$,
where P = Pressure.
T = Absolute temperature (Centigrade).
c = a constant

(4) From the vertical equilibrium of a particle, $dP = -\rho g dh$
where h = distance measured vertically, and g = acceleration due to gravity.

From the above, there immediately follows the well known result that the curvature of a ray of light passing through concentric* layers of air is:—

$$\frac{k}{\mu}\frac{d\rho}{dh}\cos a$$

or $-\frac{k\cos a}{c\mu}\frac{P}{T^2}\left(\frac{g}{c}+\frac{dT}{dh}\right)$, where a is the angle at which the ray is inclined to the concentric layers. In the rays under discussion a is small and cos a may be taken to be unity.

Consider pound, foot, second, Centigrade units.

$\mu = 1\cdot0002929$ at standard temperature and pressure.

ρ at standard temperature and pressure is $\frac{1}{12\cdot4}$ lbs. to the cubic foot.

$$\therefore k = \cdot0002929 \times 12\cdot4 = 3\cdot63\times10^{-3}$$

$$c = 3100 \text{ and } \frac{g}{c} = \frac{1}{96} = 0\cdot0104.$$

$$\text{The ncurvature} = 1\cdot17\times10^{-6}\frac{P}{T^2}\left(\cdot0104+\frac{dT}{dh}\right)$$

$$= \cdot241\,\frac{P}{T^2}\left(\cdot0104+\frac{dT}{dh}\right) \text{ seconds per foot} \quad (1)$$

Convert P into inches of mercury instead of poundals per square foot.

---

* i.e. In a vertical section the layers of equal density form concentric circles.

Then curvature $= 550 \dfrac{P}{T^2}\left(\cdot 0104 + \dfrac{dT}{dh}\right)$ seconds per foot . . (2)

T in Centigrade (absolute)

$= 1000 \dfrac{P}{T^2}\left(\cdot 0187 + \dfrac{dT}{dh}\right)$ seconds per foot . . (3)

T in Fahrenheit (absolute)

Coefficient of refraction $= \frac{1}{2}$ (Total curvature in 100 feet)

$= 27{,}000 \dfrac{P}{T^2}\left(\cdot 0104 + \dfrac{dT}{dh}\right)$ . . . . (4)

in foot, Centigrade (absolute), and inches of mercury units.

$= 50{,}000 \dfrac{P}{T^2}\left(\cdot 0187 + \dfrac{dT}{dh}\right)$ . . . . (5)

in Fahrenheit (absolute) units.

As an approximation take $P = 29'' \cdot 5$

$T = 65°F. = 523°F.$ (absolute)

Then coefficient of refraction $= 5 \cdot 4 \left(\cdot 0187 + \dfrac{dT}{dh}\right)$ . . . . (6)

$\dfrac{dT}{dh}$ being in degrees Fahrenheit per foot.

APPENDIX G

EFFECT OF THE EARTH'S ROTATION ON WATER LEVEL

As is well known, the Earth's rotation tends to deflect north or south flowing rivers towards one bank, so that the water level on that bank stands slightly higher than on the other. When crossing by the water-gauge method, it may be necessary to apply a small correction on this account. If the correction be large, the site must be considered bad, because the correction cannot be well determined.

The correction is $0 \cdot 0000023\ v\ a \sin \lambda \cos \alpha$ feet, where v is the mean velocity of the river in feet per second, a is the width of water in feet, λ is the latitude and α is the angle which the direction of flow of the river makes with the meridian. For the sign, the rule is that in the northern hemisphere the right bank is raised.

PART II

THE ERROR DUE TO REFRACTION WHEN LEVELLING UP A HILL.

M. Lallemand's formula.—When levelling under ordinary circumstances the refractions in the fore and back rays are assumed equal, and are neglected. This procedure is quite justifiable, for although there may not be even approximate equality in any particular case, the error introduced at each station is extremely small, and the errors at successive stations are not generally likely to occur with any persistence of sign. But when the line proceeds continuously along an up or down gradient the circumstances are different, and it is probable that the errors will accumulate to an appreciable extent. M. Lallemand has given* a formula for this error, depending on the air temperatures at the points where the ray meets the fore staff, instrument, and back staff. His formula has the defect that it fails if the temperature at the instrument is not intermediate between the temperatures at the staves. Such a condition may theoretically be abnormal, but in practice it has been found to be of frequent occurrence and it has not been possible to use this formula for the recently revised line from Dehra Dūn to Mussoorie. This Dehra-Mussoorie line is 18 miles long and rises continuously 4350 feet. It is consequently an extreme case of up-hill levelling. It was relevelled in 1926-27, and temperatures were measured as required for M. Lallemand's formula.

A different formula.—The curvature of a horizontal ray of light at any point depends on the vertical temperature gradient $\frac{dT}{dh}$ at that point (See Part I, Appendix F), so that before it is possible to calculate the refraction it is necessary to determine an expression for $\frac{dT}{dh}$.

For the present purpose it is assumed that $\frac{dT}{dh}$ is a linear function of h. Theoretically, some exponential relation may be more plausible, but, as will be seen later, $\frac{d^2T}{dh^2}$ is so doubtfully determined that the inclusion or exclusion of higher coefficients is of no consequence.

Let δh be the required correction to the observed heights.

Let $\frac{dT}{dh} = p + qh$, q being $\frac{d^2T}{dh^2}$.

Let the distance between the staves be $2a$ feet, and let the rise between them be $2H$ feet, the slope being uniform.

Let the slope of the ground be α, so that $\tan \alpha = \frac{H}{a}$.

Let the angle of refraction in the fore ray be Ω_1, and in the back ray Ω_2.

---

* Comptes Rendus des Séances de la Commission permanente de l' Association Géodésique Internationale, 1896.

Let s be the horizontal distance of any point from the instrument, and h its height above the ground. At the instrument assume h to be 5 feet.

From Part I, Appendix F, the curvature at any point is

$$\frac{550P}{T^2}\left(\cdot 0104 + \frac{dT}{dh}\right) \text{ seconds per foot.}$$

$$\text{whence } \Omega = \frac{1}{a}\int_0^a 550 \frac{P}{T^2}\left(\cdot 0104 + \frac{dT}{dh}\right)\left(a - s\right)ds.$$

Now $\frac{dT}{dh} = p + qh$ and $h = 5 \pm s \tan a$ + for the back ray, − for the fore ray

$$\therefore \quad \frac{dT}{dh} = \left(p + 5q\right) \pm qs \tan a$$

$$\text{and } \Omega = \frac{550P}{aT^2}\int_0^a \left\{ \left(\cdot 0104 + p + 5q\right) \pm qs \tan a \right\} \left\{ a - s \right\} ds$$

+ for Ω_2, and − for Ω_1

$$= \frac{550P}{aT^2}\left\{ \tfrac{1}{2}\left(\cdot 0104 + p + 5q\right)a^2 \pm \frac{a^3}{6}\, q \tan a \right\}$$

The height correction $\delta h = a\left(\Omega_2 - \Omega_1\right) \sin 1''$

$$= \frac{550P}{T^2} \cdot \frac{a^2 qH}{3} \sin 1''$$

$$= \frac{180P}{T^2} \cdot \frac{d^2T}{dh^2}\, a^2 H \sin 1'' \quad \text{.............(7)}$$

P in inches of mercury
T in degrees Centigrade (absolute)
h and a in feet.

Correction for the Dehra-Mussoorie line.— $\frac{d^2T}{dh^2}$ no doubt varies from place to place and from time to time, but since the correction is only of consequence if it is systematically accumulating, and since it is in any case small, computational labour may be saved by obtaining a single value for any large section of the line. In the present case the line has been divided into three equal sections, the agreement between which provides a means of judging the accuracy of the value found. Section I from Dehra to Rājpur is rather less steep than the other two.

Denoting the temperatures at the instrument and either staff by T_I and T_S respectively, and the height of the ray above the ground at these places by H_I and H_S, and assuming that isothermal layers are parallel to the ground, we have $\frac{dT}{dh}$ at a height of $\frac{H_I + H_S}{2} = \frac{T_I - T_S}{H_I - H_S}$

H_I is fairly constant (5 feet). All the observations have been classified in groups according to the values of H_S (e.g. H_S between 0 and 1 foot, 1 and 2 feet etc.,) and the mean value of $T_I - T_S$ has been taken out for each group. $\frac{T_I - T_S}{H_I - H_S}$ has then been computed, using for H_S the middle height of the group (6 inches, 1½ feet etc). The resulting values of $\frac{dT}{dh}$ are given in Table 21. No results are given for groups in which $H_I - H_S$ is between +3 and −3 feet, as the small denominator necessarily makes them inaccurate.

TABLE 21. Gradient in °C per foot.
+ indicates temperature increasing with height.

Gradient at *feet*	Section I	Section II	Section III	Mean	
2¾	−·03	−·05	−·07	−·050	−·036 at 3 feet
3¼	−·00	−·01	−·06	−·C23	
6¾	+·05	+·06	−·04	+·023	+·003 at 7 feet
7¼	+·01	−·01	−·05	−·017	

From Sections I, II and III $\frac{d^2T}{dh^2} = +\cdot011,\quad +\cdot014$ and $+\cdot005$ respectively.

Mean value of $\frac{d^2T}{dh^2} = +\cdot010$ °C per (foot)^2.

The three values of $\frac{d^2T}{dh^2}$ show little similarity beyond identity of sign, and the result can be credited with little accuracy. Some confirmation of the order of magnitude of $\frac{d^2T}{dh^2}$ can be obtained from the direct measures of the temperature gradient given in Part I, Appendix E. Tables 18 and 19 give mean values of $\frac{dT}{dh}$ at 13 feet and 31 feet, from which $\frac{d^2T}{dh^2}$ at 22 feet is found to be ·0034 and ·0053 at Dehra Dūn and Balawali respectively in Fahrenheit units, or ·002 and ·003 in Centigrade. At five feet above the ground, with which height the leveller is concerned, $\frac{d^2T}{dh^2}$ may be expected to be greater. In Part I the falling off of $\frac{dT}{dh}$ (and consequently of $\frac{d^2T}{dh^2}$) was found to be fairly proportional to $e^{-\cdot065h}$. Accepting this, we get ·005 and ·008 for $\frac{d^2T}{dh^2}$ at 5 feet, which is in good agreement with the figure now obtained.

Substituting $\frac{d^2T}{dh^2} = \cdot010$ in (7) we have,

Height correction $= 1\cdot8\ \frac{P}{T^2} \sin 1'' \Sigma a^2 H.$

Table 22 gives the data for calculating the correction.

TABLE 22

Section	$\Sigma\ a^2H$ (feet)^3	P inches	T' Centigrade (abs.)	Height Correction feet
I	$5 \cdot 80 \times 10^6$	27·1	298	·0154
II	$3 \cdot 44 \times 10^6$	25·6	297	·0089
III	$1 \cdot 77 \times 10^6$	24·2	293	·0045

The total correction is 0·029 feet. This is to be *added* to the observed height of Mussoorie above Dehra, since a positive value of $\frac{d^2 T'}{dh^2}$ results in less refraction in the fore ray than in the back, which results in the back staff reading being relatively too small, and the observed height difference being also too small.

This result may be compared with that of 0·092 feet obtained by Dr. J. de Graaff Hunter* in 1909, using Lallemand's formula and using values obtained by M. Lallemand elsewhere for what is equivalent to $\frac{d^2 T}{dh^2}$. In view of the absence of any actual observations of this essential quantity for the estimate of 1909, the agreement between ·029 and ·092 is not unsatisfactory.

The probable error due to ordinary causes in a precise line 18 miles long is ·025 feet, so that a correction of ·029 should undoubtedly be included if it could be at all correctly determined. In the present case the value deduced for $\frac{d^2 T}{dh^2}$ is believed to be so inaccurate that the correction is not worth applying. We can only be satisfied that it is not so large as to make the uncorrected result seriously inaccurate, and that it can in no way be called upon to explain the unaccounted discrepancy of one or two feet between the spirit-levelled value of the height of Mussoorie above Dehra and that found by triangulation.†

Conclusions.—The Dehra-Mussoorie line is an extreme case of up-hill levelling. Its refraction correction has been found almost negligible, from which it may at first sight be concluded that the correction will be negligible in all lines. Such a conclusion is not entirely correct, for two reasons. Firstly, formula (7) shows that the correction varies as a^2. Between Dehra and Mussoorie a averaged less than 1½ chains, while in more gently sloping country it may average 4 chains. Secondly, the Dehra-Mussoorie line gave a value of ·010 for $\frac{d^2 T}{dh^2}$. This was slightly greater than that deduced from the figures given in Part I Appendix E, but it is only one third of that used in Vol. XIX Appendix 4. It is reasonable therefore to legislate for a probable value of ·020 and to recognise that larger values may occur.

* G.T.S. Vol. XIX, Appendix 4.

† *Vide* Dr. J. de Graaff Hunter in Survey of India Professional Paper 14. Appendix

Substituting $a = 264$ feet and $\frac{d^2T}{dh^2} = \cdot 020$ in (7) with normal values of P and T, and summing the correction for consecutive rays, it is seen that $\delta h = \cdot 020 \, \Sigma \frac{H}{50}$, i.e. ·020 per total rise of 100 feet. This figure is independent of the slope provided the gradient is easy enough to allow of 4 chain shots. In the very flat slopes which persist for hundreds of miles in the plains of India, an error of this magnitude does not matter. For instance the height of Lahore, 700 feet above sea level, may be in error by 0·140 feet, which is negligible. On steeper slopes the error becomes of more consequence, culminating at a slope of 1 in 70 where an error of ·020 per 100 foot rise is equivalent to 0·015 feet per mile, an amount which becomes intolerable after a very short distance. On steeper slopes, the length of shot is necessarily reduced: at 1 in 20 only $1\frac{1}{3}$ chain shots are possible, and the error is reduced to ·005 per mile. It is a fortunate fact that on a road a slope of 1 in 70 seldom persists for long: if a hill of 100 feet or more has to be climbed, the gradient generally steepens to something like 1 in 20. Exceptions of course occur, notably when a road approaches the foot of a great range of hills. Also, long gradients of 1 in 70 may occur on railway lines. Modern high precision lines are, however, kept away from railways as far as possible.

In the very occasional places where dangerously persistent easy gradients occur, the remedy seems to lie in shortening the length of shot, rather than in making temperature readings and trying to find a correction. The following four rules indicate the circumstances under which such shortening is necessary. On the one hand they are sufficiently lax to avoid their becoming a perpetual nuisance (they will be found to require very infrequent action), and on the other hand their observance should ensure the absence of serious error in any line but one especially designed to defeat them. Definite rules are necessary for the guidance of the levellers in the field:—

Rule 1. If the rise in any section about five miles in length average less than 1 foot per station, (i.e. 50 feet in 5 miles), that section is considered flat and no action is necessary.

Note A. When taking out the average rise, any negative rises are of course numerically subtracted from the positive rises.

Note B. With shots averaging 4 chains, and $\frac{d^2T}{dh^2} = \cdot 020$, this rule allows an error of ·002 per mile, which is tolerable for considerable distances.

Rule 2. If the country is not flat as defined in rule (*1*), it is considered to have a dangerously persistent gradient after it has risen more than 50 feet, if the length of the shots (station to staff) in which the rise has occurred has averaged 3 chains or more, or 100 feet if the shots have averaged between $1\frac{1}{2}$ and 3 chains. If they have averaged less than $1\frac{1}{2}$ chains there is no limit.

Note C. This rule admits an error of ·010 feet before any action is taken.

Rule 3. Once a gradient has been found to be persistent (i. e. after it has risen 50 or 100 feet as above) the length of shot must be reduced to $1\frac{1}{2}$ chains, until such time as the gradient is reversed or so reduced that the rise per station does not average more than 1 foot with a longer shot.

Note D. $1\frac{1}{2}$ chain shots admit an error of ·005 per mile (at the worst slope). This can be tolerated for some miles.

Note E. If the gradient be reversed or reduced as above for one or more consecutive shots, the length of these shots should not be restricted, but the dangerously presistent gradient cannot be considered to have ceased until the down hill or level has continued for at least one mile (see also Note F), i. e. if the rise be resumed before one mile, the length of shot must immediately be reduced to $1\frac{1}{2}$ chains, without waiting for a second rise of 50 or 100 feet.

Note F. If after reversal the down hill gradient be such as to come under rule (2), the up grade is considered to have ceased, and the length of shot must in due course be reduced on account of the down grade.

Rule 4. The above three rules are repeated substituting the words "fall" for "rise", "down" for "up", and *vice versa.*

If, in future, temperature measurements should again be made and a correction applied, it would be better to take the temperature at three points vertically above each other near the instrument. For $\frac{d^2T}{dh^2}$ is the essential quantity which has to be measured, and it is much better determined by three such measures at one place, than by measures at three places some chains apart.

PART III

THE CORRECTION FOR STAFF LENGTH

The present method of correcting for the departure of the true length of the staff from 10 feet is to take the mean of the errors of the two staves, and to apply it to the differences of height between bench-marks. It is clear that this procedure is only strictly correct if the two staves are of exactly equal length. In view of the presence of inexplicable systematic errors in all levelling, this possible source of error has been investigated, with the result that the present method is concluded to be satisfactory, provided that the staves are somewhat carefully paired as regards equality of length.

Let the two staves be designated A and B.

Let e_A and e_B be their errors expressed as decimals of their length.

i.e. length of $A = 10\ (1 + e_A)$ feet.

Let B indicate a back staff reading, and F a fore staff reading.

Let suffixes A and B indicate the staff on which the reading is made.

Then the true correction for the length of staff

$$= e_A\ \Sigma\,(B_A - F_A) + e_B\ \Sigma\,(B_B - F_B)$$

The present form of correction $= \dfrac{e_A + e_B}{2}\ \Sigma\,(B - F)$

$$= \frac{e_A + e_B}{2}\ \Sigma\left\{(B_A - F_A) + (B_B - F_B)\right\}$$

Let E designate the true correction *minus* the present correction.

$$\text{Then } E = \frac{e_A - e_B}{2}\ \Sigma\left\{(B_A - F_A) - (B_B - F_B)\right\}$$

$$= \frac{e_A - e_B}{2}\ \Sigma\,(R_A - R_B), \quad . \quad . \quad . \quad . \quad (1)$$

where R_A is the rise in height of the line of sight when the instrument is moved and A staff remains stationary. Similarly R_B is the rise when B remains stationary.

The sign of small corrections requires to be carefully stated. If e_A be defined by the relation "Actual length of staff $= 10\ (1 + e_A)$", E must be added to the height of the fore bench-mark above that of the back bench-mark.

$e_A - e_B$ may be of constant sign throughout the season. On the other hand the sign of $R_A - R_B$ is truly casual, as is therefore the sign of their product, which is consequently amenable to the laws of probability. If n be the number of times the instrument is set up, K the probable value of $R_A - R_B$ for two consecutive moves, and P. E. the probable value of E, then

$$\text{P. E.} = \frac{e_A - e_B}{2} K \sqrt{\frac{n}{2}} \quad \ldots \ldots (2)$$

Our present field operations supply all the data necessary for the evaluation of this correction in its form (1), and unless we can be satisfied that it is negligibly small it should undoubtedly be computed and applied. Formula (2) provides the means of estimating its magnitude. By pairing staves, the magnitude of $e_A - e_B$ can be limited. Given a fairly large stock of staves, they can be so paired that it does not exceed ·0002 at the beginning of the season. But such close agreement cannot be maintained, and examination of previous years' records shows that a difference of 0·0004 may occasionally occur and persist for some time, even if the staves be paired to 0·0002 at the start. $e_A - e_B$ must therefore be taken as 0·0004. The estimation of n presents no difficulty. Table 23 shows some actual values.

K is not an easy thing to guess, but typical values can be obtained by computation from past records. Table 23 gives the results. Each entry is deduced from about 50 actual values of $R_A - R_B$ taken at random along the line, using the formula $K = \frac{\cdot 6745}{\sqrt{n-1}} \sqrt{\Sigma (R_A - R_B)^2}$.

$R_A - R_B$ is of course zero on a surface which is absolutely flat or on a constant grade, provided the instrument is set up at constant height. Along railway lines, or along good roads in very flat country, these conditions are closely realised and the table shows that K is between 1·0 feet and 1·5 feet. When working on a long steep slope K is also small, because the level is deliberately so placed that the rise shall always be nearly the maximum possible amount of 8 or 9 feet. This point is of importance because in such places n may be abnormally large. It is clear however, that no danger arises, and exceptionally large values of n need not be considered. Sharply undulating country is the most unfavourable. The last column of Table 23 gives the probable error generated in 100 miles by the method of applying staff correction, for 5 recent lines of levelling. It is a truly accidental error, in no way systematic. The allowable accidental probable error in 100 miles is ·041 of a foot, a figure which is not much improved upon in practice. The largest value of P. E. in the table is 0·013 ft., which is negligible. With well paired staves, the average value in all lines will be about one third of this.

It is concluded that the present method of applying the correction is satisfactory, but that staves should be paired to within 0·0004 (i.e. 0·004 feet). If they start the season within 0·002 feet, the above limit is not likely to be exceeded. If it should be exceeded, the desirability of applying the correction given by formula (1) should be considered. If a sample be taken and K be found small, the correction will probably be of no consequence.

TABLE 23—*Error in present method of applying staff correction*

Line and Date	Length	Range of height	n = number of stations per 100 miles	K	Largest actual error	P.E. after 100 miles ‡	Nature of line
	miles	*feet*		*feet*	*feet*	*feet*	
Lahore-Feroze-pore-Daroli 1913-14 & 1919-20	85	70	1,400	1·40		·007	Along a road in the flat plains of the Punjab.
Sehwān-Kotri 1920-21	88	50	850	1·27		·005	Along a railway in Sind.
Rājkot-Porbandar 1926-27	132	600	1,250	1·15	·012*	·006	Along road and railway. Part undulating & part flat.
Raichūr-Bāgalkot 1914-15	124	600	1,300	2·62		·013	Along a road in undulating country in the Deccan.
Rājpur-Mussoorie 1926-27	13½	3,200	4,500	1·45	·001†	·013	Along road & pony track on a steep hill.

* $e_A - e_B$ averaged ·0007

† $e_A - e_B$ averaged ·0001

‡ assuming $e_A - e_B =$ ·0004

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